Bond immunization - practical example

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Bond immunization

- A bond portfolio's value in the future depends on the interest-rate structure prevailing up to and including the date at which the portfolio is liquidated.
- If a portfolio has the same payoff at some specific future date, no matter what interest rate structure prevails, then it is said to be **immunized**.

Basic example

- A firm has a known future obligation Q. The discounted value of its obligation is $V_0 = \frac{Q}{(1+r)^N}$ where r is the appropriate discount rate.
- Suppose that this future obligation is hedged by a bond held by the firm, i.e. the value of the bond V_B equals the discounted value of future obligations V_0: $V_B = \sum_{t=1}^{M} \frac{P_t}{(1+r)^t}$ where P_t are the stream of anticipated payments made by the bond.
- If the underlying interest rate r changes to $r + \Delta r$: -$V_0 + \Delta V_0 \approx V_0 + \frac{dV_0}{dr}\Delta r = V_0 + r[-\frac{NQ}{(1+r)^{N+1}}]$ -$V_B + \Delta V_B \approx V_B + \frac{dV_B}{dr}\Delta r = V_B + \Delta r \sum_{t=1}^{N} \frac{-tP_t}{(1+r)^{t+1}}$
- Portfolio is **immunized** if $V_0 + \Delta V_0 \approx V_B + \Delta V_B$, i.e. if $\frac{1}{V_B} \sum_{t=1}^{M} \frac{tP_t}{(1+r)^t} = N$

Basic example

- To improve the immunization of the bond portfolio to changes in the yield, not only the first derivative (**duration**) but also the second derivative (**convexity**) should match.

- We need three bonds in order to calculate a portfolio of bonds whose duration and whose second duration derivative are exactly equal to those of the liability.
- **Solution**: invest proportions (weights) w_1, w_2, w_3 in bonds 1, 2 and 3 such that:
  * the portfolio is totally invested: $w_1 + w_2 + w_3 = 1$
  * the portfolio duration is matched: $w_1 D_1 + w_2 D_2 + w_3 D_3 = D_{liability}$
  * the portfolio convexity is matched: $w_1 C_1 + w_2 C_2 + w_3 C_3 = C_{liability}$

Basic example

- In matrix form:

$$\begin{bmatrix} 1 & 1 & 1 \\ D_1 & D_2 & D_3 \\ C_1 & C_2 & C_3 \end{bmatrix} \cdot \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ D_{liability} \\ C_{liability} \end{bmatrix}$$

- **Solution**:

$$\begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ D_1 & D_2 & D_3 \\ C_1 & C_2 & C_3 \end{bmatrix}^{-1} \cdot \begin{bmatrix} 1 \\ D_{liability} \\ C_{liability} \end{bmatrix}$$

Basic example

- In excel: `http://35.180.63.47/BondImmunization.xlsm`